Finite-size corrections in Lyapunov Spectra for Band Random Matrices

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Abstract

The transfer matrix method is applied to quasi one-dimensional and one-dimensional disordered systems with long-range interactions described by band random matrices. We investigate the convergence properties of the whole Lyapunov spectra of finite samples as a function of the bandwidth and of the sample length. Two different scaling laws are found at the maximal and minimal Lyapunov exponents.

I. INTRODUCTION

Lyapunov exponents represent one of the main tools in the study of both disordered and dynamical systems. In the former case, they arise from the application of the transfer matrix method and allow determining, e.g., the localization length, which corresponds to the inverse of the minimum positive Lyapunov exponent. In the latter case, starting from the evolution in the tangent space, one is interested in determining, among other quantities, the Kolmogorov-Sinai entropy that can be expressed as the sum of the positive exponents.

Recently, a growing attention has been devoted to the study of high-dimensional systems such as coupled maps, chains of nonlinear oscillators, dynamical models with delayed feedback, disordered systems in two and three dimensions, and 1-dimensional models with long range interaction. In all these cases, the so-called Lyapunov spectrum is defined as the sequence of the Lyapunov exponents γ_i (ordered for increasing/decreasing size) represented as a function of i/D, where D is the total number of exponents. Many numerical simulations and analytical arguments indicate the existence of a limit spectrum for $D \to \infty$ (see, e.g., [1] and references therein). In contrast to the largest and smallest exponents, for which some rigorous mathematical results have been obtained, the properties of the "bulk" of the Lyapunov spectrum are less understood.

In Ref. [2], the scaling properties of the Lyapunov spectrum have been studied in disordered systems described by infinite Band Random Matrices (BRM). Such matrices have been extensively investigated in connection with 1-dimensional Anderson-type models with long range random hopping as well as with quasi 1-dimensional thin wires (see e.g. [3] and references therein). In particular, the scaling properties of the eigenfunction localization lengths proved to be in accordance with the predictions [4,5] based on results for dynamical quantum models that are strongly chaotic in the classical limit.

Since any disordered sample used in practical applications is finite, it is useful not only to study the asymptotic value of the Lyapunov exponents but also the so-called effective exponents, i.e. the exponents actually observed in samples of finite length N. Moreover, information about the statistical properties of effective Lyapunov exponents can shed further light on, e.g., the fluctuations of the conductance in the metallic regime [6–10].

In this paper, we investigate the convergence properties of the effective Lyapunov exponents of finite BRMs associated with finite samples embedded in an otherwise perfectly ordered lattice. Our investigation suggests that different parts of the spectrum exhibit different convergence properties. This is particularly clear from our data for the maximal Lyapunov exponent as compared to the bulk of the spectrum.

The outline of the paper is as follows. In Sec. 2 the connection between the Hamiltonian band matrix model and the conductance of disordered samples is summarized with the purpose of introducing the appropriate Lyapunov exponents. In Sec. 3, we recall some known results of scaling theory in similar problems: they will represent the starting point for the numerical investigation carried on in the following Sec. 4. Some final comments and conclusions are presented in the last Sec. 5.

II. THE MODEL

The general model describing quasi 1-dimensional or 1-dimensional systems with long range hopping is given by the Schrödinger equation

$$i\,\frac{dc_n(t)}{dt} = \sum_{m=n-b}^{n+b} H_{n,m}c_m(t) \quad , \tag{1}$$

where $c_n(t)$ is the probability amplitude for an electron to be at site n and $H_{n,m}$ is a symmetric band random matrix. Specifically, the entries of $H_{n,m}$ are independent Gaussian variables with zero mean and variance $\sigma^2 = 1 + \delta_{n,m}$ for $|n-m| \leq b$, while the matrix elements outside the band are all set equal zero. In 1-dimensional geometry, the parameter b defines the hopping range between neighboring sites, while in the quasi 1-dimensional interpretation, this parameter has the meaning of the number of transverse channels for the scattering waves along a thin wire [3].

The insertion of a disordered sample of length N into a perfectly ordered lattice requires the definition of two proper leads at the extrema of the sample. At variance with the standard Anderson model, where only nearest-neighbor couplings are present, the long-range hopping terms in our model allow some freedom in the structure of the ordered leads. A reasonable choice consists in assuming a band structure in the ordered lattice with the same width b and the hopping elements $H_{n,m}$ set all equal to U (for the sake of simplicity we choose $U = 1$), while the hopping amplitudes coupling the leads with sample-sites are randomly chosen with the same distribution as in the core of the sample (the intermediate regions connecting the samples with the leads will be hereafter called "contacts": they extend over b sites). As an example, we show below the Hamiltonian structure for $b = 2$ (asterisks mark random elements)

$$\begin{array}{cccccccccccc}
\ldots & \ldots & 1 & 1 & & & & & & & & \\
1 & 1 & 1 & 1 & * & & & & & & & \\
 & 1 & 1 & 1 & * & * & & & & & & \\
 & & * & * & * & * & \ldots & & & & & \\
 & & & & & \ldots & * & * & * & * & & \\
 & & & & & & * & * & 1 & 1 & 1 & \\
 & & & & & & & * & 1 & 1 & 1 & \ldots
\end{array}$$

The eigenvalue equation is obtained by inserting the standard Ansatz $c_n(t) = \exp(-iEt)\psi_n$ in Eq. (1). The resulting equation can be casted in the form of a $2b$-dimensional linear map along the spatial direction,

$$\Psi(n+1) = T(n)\Psi(n) \;, \tag{2}$$

where $\Psi_i(n) \equiv \psi_{n+b-i}$ and the matrix $T(n)$ is defined as follows,

$$\begin{aligned}
&[T(n)]_{1,j} = \frac{1}{H_{n,n+b}}\left(\delta_{j,b}E - H_{n,n+b-j}\right) \\
&[T(n)]_{i,j} = \delta_{i-1,j}; \qquad [T(n)]_{i,2b} = 0; \qquad 2 \leq i \leq 2b
\end{aligned} \tag{3}$$

In this picture, an eigenstate of Eq. (1) can be treated as a "trajectory" of the random map (3) and its localization properties are determined by the Lyapunov exponents.

In a previous paper [2] we investigated the shape of the Lyapunov spectrum in the limit of infinitely extended disordered samples. Here, since we are interested in finite samples of size N, one should introduce the transfer matrix $T = \prod_{n=1}^{N} T(n)$, which couples two opposite leads. As was shown in [6], the matrix T satisfies the following property, not shared by the single matrix $T(n)$,

$$T^t \Sigma T = \Sigma \quad , \qquad \Sigma = \begin{pmatrix} 0 & S \\ -S^t & 0 \end{pmatrix} \tag{4}$$

where S is a lower triangular matrix of size b, with $S_{ij} = 1$, $i \geq j$. In fact, this property corresponds to the flux conservation in the scattering process of a wave through the sample.

It is convenient to describe the scattering states in terms of eigenfunctions of the free dynamics occurring in the ordered region. The eigenvalues of the corresponding Hamiltonian, defined by setting all random elements equal to 1, are

$$E(p) = 1 + 2\cos p + \ldots 2\cos(bp) = \frac{\sin[(2b+1)(p/2)]}{\sin(p/2)}, \qquad p \in (-\pi, \pi) \tag{5}$$

while the corresponding eigenvectors are

$$\psi_n(p) = \frac{1}{\sqrt{2\pi}} e^{\pm inp} \ . \tag{6}$$

For any fixed energy value $\tilde{E}$, there are $\nu \leq b$ pairs of opposite real solutions of the equation $E(p_k) = \tilde{E}$. Each pair corresponds to an open channel, or propagation mode, sustaining waves with opposite velocities. In this paper, we limit ourselves to study the case $E = 0$, when all channels are open, i.e. $\nu = b$, and the admissible momenta are equispaced. In what follows we shall assume that the momenta p_k are ordered in such a way that positive velocities correspond to the first b elements,

$$p_1,\, p_2,\, \ldots p_b,\, -p_1,\, \ldots - p_b \ ,\ p_k = (-1)^k \frac{2\pi k}{2b+1} \ ,\ k = 1, \ldots, b \ . \tag{7}$$

In fact, one can see that the corresponding velocities $v(p) = dE/dp$ for $E = 0$ are given by

$$v_k = \frac{2b+1}{2\sin[\frac{\pi k}{2b+1}]} \ ,\ v_{k+b} = -v_k \ ,\ k = 1, \ldots, b \ . \tag{8}$$

Instead of defining the initial state in the scattering process in terms of the probability amplitude in $2b$ consecutive sites (as required by the standard representation of the vector $\Psi(n)$), one can refer to the $2b$ amplitudes of the plane waves sustained by the ordered lattice. It can be easily checked that the transformation to pass from the momentum to the position representation is defined by the matrix

$$[U(n)]_{j,k} = \exp\Big(i(j+n)p_k\Big) \ , \tag{9}$$

so that the scattering matrix can be written as

$$M = Z^{-N} U(0)^{-1} T U(0) \tag{10}$$

where Z is a diagonal matrix the entries of which, $Z_{j,j} = e^{ip_j}$, account for the phase difference between the sites $n = 1$ and $n = N$. The matrix M has an almost symplectic structure; indeed, it satisfies the relation $M^\dagger V M = V$, where V is a diagonal matrix with $V_{j,j} = v_j$ in the same order as before [6].

For the determination of the conductance, one needs to introduce the matrix F connecting flux amplitudes,

$$F = \Gamma M \Gamma^{-1} \ , \tag{11}$$

where the diagonal $2b \times 2b$ matrix Γ is defined as $\Gamma_{i,j} = \delta_{i,j}\sqrt{|v_i|}$. The above transformation is equivalent to the normalization of the scattering matrix and it takes into account that the waves propagate with different velocities in different open channels.

The transfer matrix F connecting the incoming with the outgoing flux amplitudes in the various channels turns out to be symplectic as it satisfies the relation

$$F^\dagger \sigma_3 F = \sigma_3 \quad , \tag{12}$$

where σ_3 is a generalized Pauli σ_z matrix,

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \tag{13}$$

and 1 denotes a $b \times b$ identity matrix. Notice that condition (12) follows essentially from the flux conservation, i.e. from the unitarity of the scattering matrix.

The matrix F is the key ingredient for the determination of the conductivity from the Landauer formula (see, e.g., [7] for a general derivation and [10] for an application in the specific case of BRMs). More precisely, it is necessary to compute the Lyapunov exponents

$$\gamma_i(N) = \frac{1}{2} \ln m_i(N) \quad , \tag{14}$$

where $m_i(N)$ denote the (real) eigenvalues of the matrix $F^\dagger F$. The Lyapunov exponents will be conventionally ordered from the maximum to the minimum one, namely $\gamma_1(N) > \gamma_2(N) > \ldots > \gamma_i(N) > \ldots > \gamma_{2b}(N)$. Because of the symplectic structure of F, the exponents are arranged in b pairs with opposite values ($\gamma_i(\nu) = -\gamma_{2b-i+1}(\nu)$); for this reason, in the following, we will always report only the positive exponents (i.e. $i \leq b$).

In the limit of infinitely long samples ($N \to \infty$), the ergodic multiplicative theorem [11] ensures that the statistical fluctuations of the effective Lyapunov exponents vanish: in fact, the quantities $\gamma_i(\infty)$'s are, by definition, the Lyapunov exponents of the infinite product of matrices composing F. Moreover, in the limit $N \to \infty$, the effect of the similarity transformations involved in the definition (10,11) of F becomes negligible, so that the values of $\gamma_i(N)$ converge also to the Lyapunov exponents of the matrix T. However, as long as one deals with finite samples, the effective Lyapunov exponents $\gamma_i(N)$ depend on the realization of the disorder. It is therefore convenient to average $\gamma_i(N)$ over the ensemble of all possible realizations. In order not to overload the notations, this average will be always understood.

III. SCALING BEHAVIOUR

Let us first discuss the scaling behaviour of the Lyapunov exponents for the Anderson model. In fact, the transfer matrix approach reveals a clear analogy between the physical problem considered in the present paper and the Anderson localization in a stripe. Indeed, the band-width b plays the same role as the strip-width L_t in the sense that both define the number of possible channels for electronic conductance. It is important to note, however, that for the analogy to be kept as strict as possible, one must assume that the Lyapunov exponents are measured in units of the interaction range (or, equivalently, in mean free paths), i.e. by referring to the lattice spacing in the Anderson problem and to b in the present case. This feature was already noticed in [2], where it was pointed out that the Lyapunov spectra of BRM, measured in natural units, scale as $1/b$.

One cannot straightforwardly apply to the present case the single-channel scaling theory to infer localization properties for different disorder amplitudes and "transversal widths", since there is no proper localization length in the thermodynamic limit $b \to \infty$. In fact, while it is conjectured that the minimum Lyapunov exponent is finite in the limit of infinitely large stripes ($L_t \to \infty$) in the Anderson problem (in the insulating regime), it vanishes as $1/b$ for band-random matrices, even using the appropriate spatial scale. Let us indeed recall that the localization length for the eigenfunctions of energy E is $l_\infty(E) = (2b^2/3)[1 - (E^2/(8b))]$ [3], i.e. it diverges as b^2 in the whole energy range.

Another feature of the scaling behaviour that has been investigated in the Anderson problem concerns the dependence of the Lyapunov exponents on the sample length for fixed transversal width L_t. In Ref. [9] it has been found that the scale dependence for the Anderson model has the form

$$\gamma_j(N) = \gamma_j(\infty) F_j(\gamma_j(\infty) N) \quad , \tag{15}$$

independently of the disorder amplitude. In the context of BRMs, the above relation is somewhat trivial, since the amplitude of the disorder can be scaled out. This is immediately realized by noticing that the elements of the transfer matrices involve only ratios of the disorder terms (apart from the energy term which is the only contribution that need being appropriately rescaled), thus revealing that their absolute scale is irrelevant.

Meanwhile, a different approach has been suggested to describe scaling properties in BRMs without leads. Such a procedure passes through the introduction of the generalized localization lengths $l_q(N)$ [5] of a generic eigenvector,

$$l_q(N) = \exp\langle \mathcal{H}_q \rangle \tag{16}$$

where $\mathcal{H}_q$ is defined as

$$\mathcal{H}_q = \frac{1}{1-q} \ln P_q; \;\; P_q = \sum_{n=1}^{N} |\psi_n|^{2q}; \;\; q \geq 2 \qquad \mathcal{H}_1 = -\sum_n |\psi_n|^2 \ln(|\psi_n|^2) \tag{17}$$

and ψ_n is the nth component of an eigenvector of the matrix. The average of $\mathcal{H}_q$ in Eq. (16) is performed over disorder and over the eigenstates corresponding to energies within a pre-specified window. It was numerically shown in [5] and analytically proved in [3] that the

rescaled localization length $l_q(N)/l_q^{GOE}(N)$, where l_q^{GOE} corresponds to full random matrices, depends only on the ratio l_∞/N. More detailed analytical studies [3] have revealed that the scaling behavior for $l_q(N)$ is very close to the form

$$l_q^{-1}(N) = l_q^{-1}(\infty) + C_q/N \quad , \tag{18}$$

for $q \neq 2$, while it holds exactly for $q = 2$. Notice that, in the latter case, the localization length $l_2(N)$ is related to the inverse participation ratio which has the simple physical meaning of the probability for a quantum particle to return to the initial position after infinite time. The second term in the r.h.s. of expression (18) represents the normalization factor $l_q^{GOE}(N)$: it was found both numerically and analytically that the coefficient C_q is always positive and independent of N. The positiveness of C_q indicates that the finite-length estimates of the localization length converge to its asymptotic value $l_q(\infty)$ from above.

Let us finally mention that the scaling relation Eq. (18) appears to be rather general as revealed by numerical simulations performed in many other models like the Kicked Rotator [4,12], 1-dimensional Anderson, Lloyd [13], and 1-dimensional dimer models [14,15].

IV. NUMERICAL ANALYSIS

A. The method

As is mentioned in Sec. II, the conductance of a finite sample can be determined from the eigenvalues of $F^\dagger F$. For this reason, here we study the convergence properties of the latter quantities by varying the sample-size N. The standard technique for the determination of the eigenvalues runs into troubles already for relatively short samples because of numerical inaccuracies due to the small denominators in Eq. (3). Such a difficulty can be overcome by adapting a specific algorithm for the computation of the Lyapunov exponents of an infinite product of matrices [16]. In fact, given a finite sample of length N (and the corresponding matrix F defined as in Eq. (11)), one can formally construct the following infinite product of matrices

$$\ldots F^\dagger F \ldots F^\dagger F \ldots F^\dagger F \ldots \quad . \tag{19}$$

Such a sequence can be recursively applied to b independent vectors, orthonormalizing them every single step. Accordingly, one finds b Lyapunov exponents that are nothing but the logarithms of the (real) eigenvalues of $F^\dagger F$. The advantage of this procedure over the standard diagonalizations methods is that the orthomormalization can be implemented for all intermediate steps in the construction of F (i.e. multiplication by the transfer matrices T and application of the similarity transformations). This approach proved already its effectiveness in the study of the Anderson problem [17].

In practice, the number M of "replicas" of $F^\dagger F$ in (19) is finite: we have chosen M so as to guarantee at least an accuracy 10^{-4} for all the Lyapunov exponents, which means $M > 2500$.

B. Convergence to the asymptotic limit

The main goal of this subsection is the study of the convergence of $\gamma_i(b,N)$ towards the set of self-averaged Lyapunov exponents $\gamma_i(b,N\to\infty)$ as a function of the sample length N and of the bandwidth b.

We have already seen that BRMs have somewhat peculiar properties which make problematic the application of the standard scaling theory to Lyapunov exponents. A problem which, instead, makes perfectly sense also in the context of BRM, is the attempt to combine in a single relation the dependence of the Lyapunov exponents on the sample length N and the "transversal" width b. This is an issue that has not yet received a clear answer in the 2- and 3-dimensional Anderson problem [9].

The natural starting point is represented by Eq. (18), which gives the localization length as measured directly from the eigenfunctions of the Hamiltonian. By recalling the b^2-dependence of $l_q(\infty)$, one realizes that the finite-size corrections do depend only on a scaling parameter, namely N/b^2. Accordingly, one could expect that the proper scaling relation for the Lyapunov exponents is of the type

$$\gamma_i(b,N)/\gamma_i(b,\infty) = f(i/b, N/b^2) \quad , \tag{20}$$

where we have added an i/b dependence to account for possible differences exhibited by the various exponents. However, a careful analysis of our data definitely rules out such a possibility, so that one needs to modify the above Ansatz in a more suitable manner. After many different attempts to find the correct scaling dependence on b and N, we have come to the conclusion that the most convincing and yet simple scaling hypothesis is

$$\gamma_i(b,N)/\gamma_i(b,\infty) = f(\gamma_i(b,\infty)Nb^\alpha) \quad . \tag{21}$$

with α as some function of the ratio i/b. An effective test of the scaling relation can be made after subtracting the asymptotic value 1 from both sides of Eq. (21), i.e. by studying the behaviour of the difference

$$\delta\gamma_i(b,N) \equiv 1 - \gamma_i(b,N)/\gamma_i(b,\infty) \quad . \tag{22}$$

First, it is necessary to determine the asymptotic Lyapunov exponents ($N\to\infty$) for different bandwidths. In this limit, the "contacts" between the the ordered regions and the sample play no role as well as the similarity transformations involved in the definition of F. Accordingly, one can get rid of most of the technical difficulties and determine $\gamma_i(b,\infty)$ by resorting to the usual transfer matrix approach as implemented in Ref. [2]. The results reported in Fig. 1 indicate a convergence of the type $1/b$ in the bulk of the spectrum.

Once the asymptotic values of γ_i have been determined, we have plotted the finite-size correction $\delta\gamma_i$ versus the rescaled sample length $m = \gamma_i(b,\infty)Nb^\alpha$ for different choices of α. In all cases, we find that $\delta\gamma_i$ is positive, indicating that the convergence to the asymptotic values is from below. This striking difference with the behaviour of the directly computed localization length (see Eq. (18)) is the clearest indication that the influence of the leads and the type of contacts results in strong finite-size corrections.

The best data collapses obtained for $i/b = 0.3$, 0.5 and 0.9 are reported in Fig. 2. In all cases, $f(m)$ turns out to be essentially equal to $1 + A/m$ with the value of A depending very

little on i/b ($A \approx 1.1$). As anticipated in Eq. (21), the main dependence on i/b is contained in the exponent α which appears to change linearly with i

$$\alpha = i/b - 0.3 \tag{23}$$

Notice that the necessity to introduce a different scaling Ansatz, Eq. (21), in substitution of Eq. (20) is contained precisely in the above expression of α. In fact, only the equality $\alpha = -1$ for $0 < i/b < 1$ would reconcile the two scaling functions. A partial justification for this result comes from the observation that the bulk of the Lyapunov spectra scales in a different way from the minimum value (to which Eq. (20) refers). However, this is not enough to understand why the various parts of the Lyapunov spectrum exhibit different convergence properties.

A clear exception to relationship (23) is found by analyzing the behavior of the maximal exponent. This is not a surprise, since in Ref. [2] it was already noticed the existence of a "phase-transition" in the Lyapunov spectrum occurring approximately at $i/b = 0.1$. This is illustrated in Fig. 3 where $\chi = \gamma_i i$ is plotted versus i/b revealing an incipient discontinuity in the derivative of the spectrum. It is thus reasonable that different convergence properties are observed above and below $i/b \approx 0.1$. Actually, we find that the convergence is of the type $1/N$ in both cases, but the value of α is -1 for the maximum exponent, as seen in Fig. 4 (it should be recalled that γ_1 exhibits a different scaling behaviour from that in the bulk of the spectrum, being independent of b).

The behaviour of the minimal exponent is another important test, but before discussing this case, we would like to stress that the scaling of γ_b as $1/b^2$ makes the numerical computations much more difficult: in fact, it is very hard to get rid of statistical fluctuations when b becomes relatively large. An indication of this difficulty is already revealed by the comparison of the best overlaps obtained in the various cases: increasing fluctuations are detected upon increasing i/b, testifying to the importance of statistical fluctuations (this is particularly visible in Fig. 2c, i.e. for $i/b = 0.9$). Nevertheless, one can see in Fig. 5 that the data collapse is still not bad by assuming $\alpha = 0.7$, i.e. the value predicted by the linear law (23). Consistency with Eq. (20) would require $\alpha = 0$, which gives definitely a much worse overlap of the various curves and has to be, therefore, ruled out. Moreover, a regression of the various curves corresponding to different values of b seems to suggest that the convergence to the asymptotic value in this last case is slower than $1/N$, but it is not clear whether this is an artifact due to a lack of sufficient statistics or whether it is a finite-size (N) effect.

After having presented a possible unified description of the convergence properties of "finite-length" Lyapunov exponents, it is worth discussing the origin of the corrections expressed by f in Eq. (21). One reason for these corrections is the presence of the "contacts". Since the "contacts" are less disordered compared to the bulk, one can expect that they are characterized by a different, smaller, growth rate. The first prediction of this argument is a negative sign for the correction, i.e. a convergence from below, as it is indeed observed. This accounts for the main difference found with the direct investigation of the localization properties. Moreover, if this were the only source of corrections, and if there were no boundary effects between the leads and the sample, one should conclude that the relative correction must be proportional to b/N i.e. proportional to the ratio between the length of the leads and the length of the sample. This statement is equivalent to saying that Eq. (21) holds with

$\alpha = -1$ for the maximal Lyapunov exponent. As we have already seen, this prediction is perfectly confirmed. Furthermore, direct numerical simulations made to compute separately the growth rate in the contacts and in the sample do reveal that the former contribution is half of the latter almost independently of b. This proves directly the correctness of our simple conjecture.

The same argument, applied to the rest of the spectrum, would imply that the correction is still of the order of b/N which means that the α value in Eq. (21) is zero independently of i/b (except for the minimum). Since a strictly positive α is found, instead, for $i/b > 0.3$, this means that the actual correction is smaller than expected from the above argument. We can only give a qualitative explanation for the discrepancy: as long as i/b is strictly larger than $1/b$ (in the asymptotic limit $b \to \infty$), the correspoding growth rate γ_i is of the order of $1/b$ (except for the limit case of the minimum exponent) so that the contribution to the expansion observed in the "contacts" is not uncoupled to the expansion in the rest of the sample and this makes the subdivision of the entire sample into a bulk and two "contacts" less defined. Moreover, we should add that even in the absence of the leads, finite-size corrections must be present and, as far as we know, there are no theoretical predictions about this kind of corrections apart from the maximal exponent [18].

V. CONCLUSIONS AND PERSPECTIVES

We have investigated the finite-length Lyapunov spectra of symmetric Band Random Matrices describing quasi 1-dimensional and 1-dimensional disordered systems with long-range interactions. Our main goal was to investigate the scaling properties of Lyapunov spectra upon changing both the bandwdith b and the sample-size N. To our knowledge, the only example in the literature where a similar question has been addressed is Ref. [9], where the authors have commented about the way to combine the scaling behaviour with the sample length and with the strip width. However, their conjectures are not supported by numerical analysis. Here, instead, a detailed numerical investiagation suggests that all the convergence properties can be described in terms of a scaling parameter b^η/N with η depending on i/b (here, we refer to η instead of α to understand the dependence of $\gamma_i(b, \infty)$ on b as well). Such a behavior does contrast with our expectations a priori of a b^2/N dependence based on the results of a direct analysis of the localization properties.

Even more striking is, in contrast to the 2-dimensional Anderson model, the negative sign of the finite-size correction terms. These results can be partly attributed to the influcence of the "contacts" connecting the ordered leads with the disordered sample. However, while the b/N dependence for the maximal exponent is also supported by a simple theoretical argument, the same cannot be said for the other exponents. Nevertheless, we wish to recall that other combinations of simple functions provide definitely less accurate descriptions of the observed data. On the other hand, we cannot exclude that the relative "smallness" of the values of b and N accessible to a numerical analysis, masks a dependence different from that conjectured in Eq. (21). We can only stress that the scaling Ansatz is chosen as by far as the simplest one in a set of even less convincing alternatives. In other words, our work represents an instance of the application of the Occam's razor. Thus, some theoretical progress is needed to shed further light on this problem. This is particularly true for the minimal exponent which is the most delicate one to be numerically determined.

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FIGURES

FIG. 1. Convergence of the asymptotic Lyapunov exponents $\gamma_i(b,\infty)$ upon increasing b. The three data sets refer to $i/b = 0.3$ (plusses), 0.5 (diamonds), and 0.7 (circles). In all cases, the difference $\delta\gamma_i = \gamma_i(b,\infty) - \gamma_i(\infty,\infty)$ is plotted versus b. The straight lines follow from a best fit: their slopes are always close to -1 (with a few percent deviations).

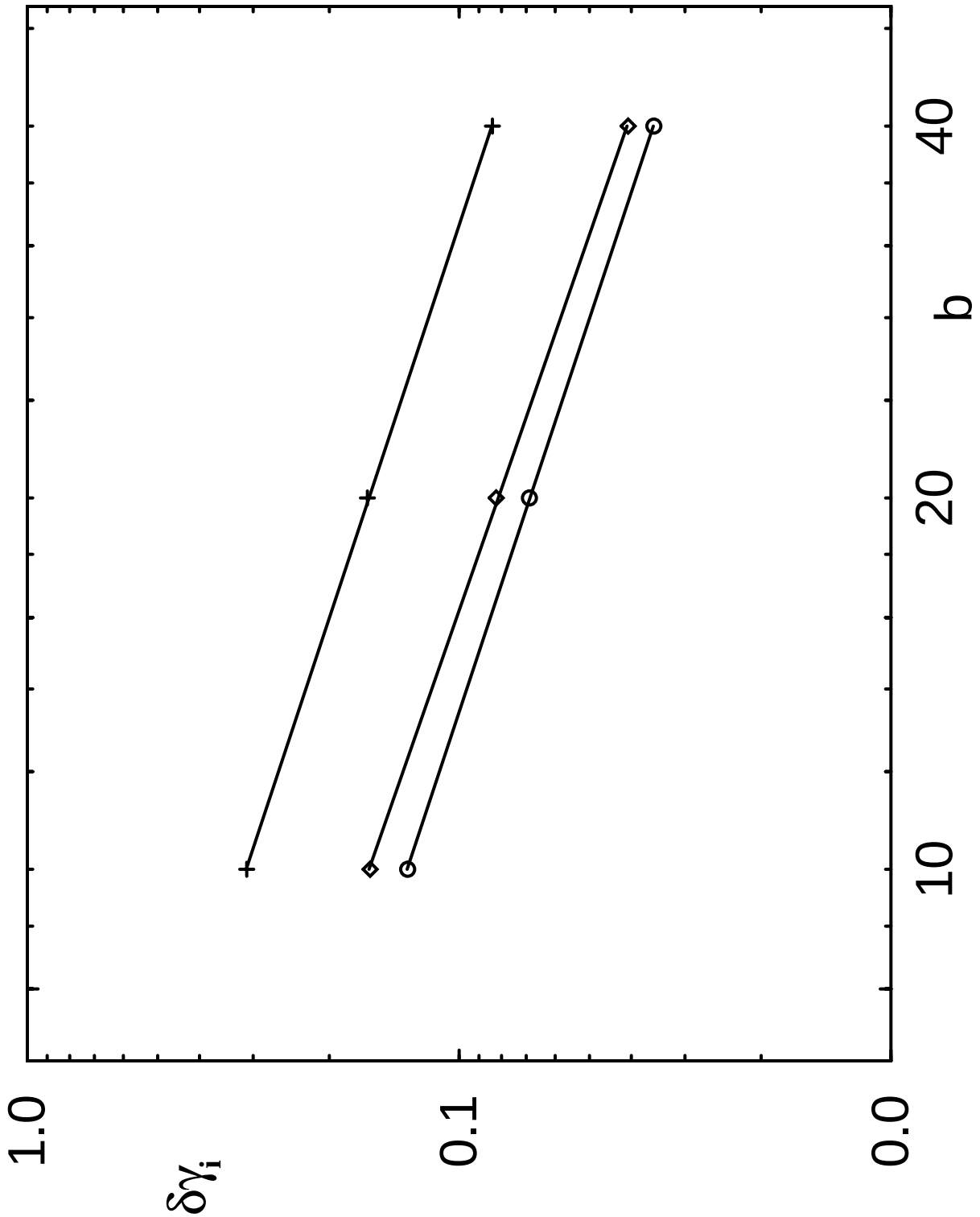
FIG. 2. Rescaled finite-size corrections $\delta\gamma_i$ to the Lyapunov exponents versus the rescaled length of the sample size m: circles correspond to $b = 10$, diamonds to $b = 20$ and plusses to $b = 40$; (a), (b), and (c) correspond to $i/b = 0.3$, 0.5 and 0.9, respectively. The straight lines are the best fits. The deviations of the resulting slope from -1 is approximately 3% in all cases.

FIG. 3. Lyapunov spectrum as determined for $b = 10$ (circles), 20 (squares), 30 (diamonds), and 40 (triangles).

FIG. 4. Finite-size corrections of the maximum Lyapunov exponent, using the same representation as in Fig. 2. The slope of the straight line is -1.

FIG. 5. Finite-size corrections for the minimum Lyapunov exponent, using the same representation as in Fig. 2. The straight line with slope -1 is drawn for reference.

Fig.1 Kottos et al.



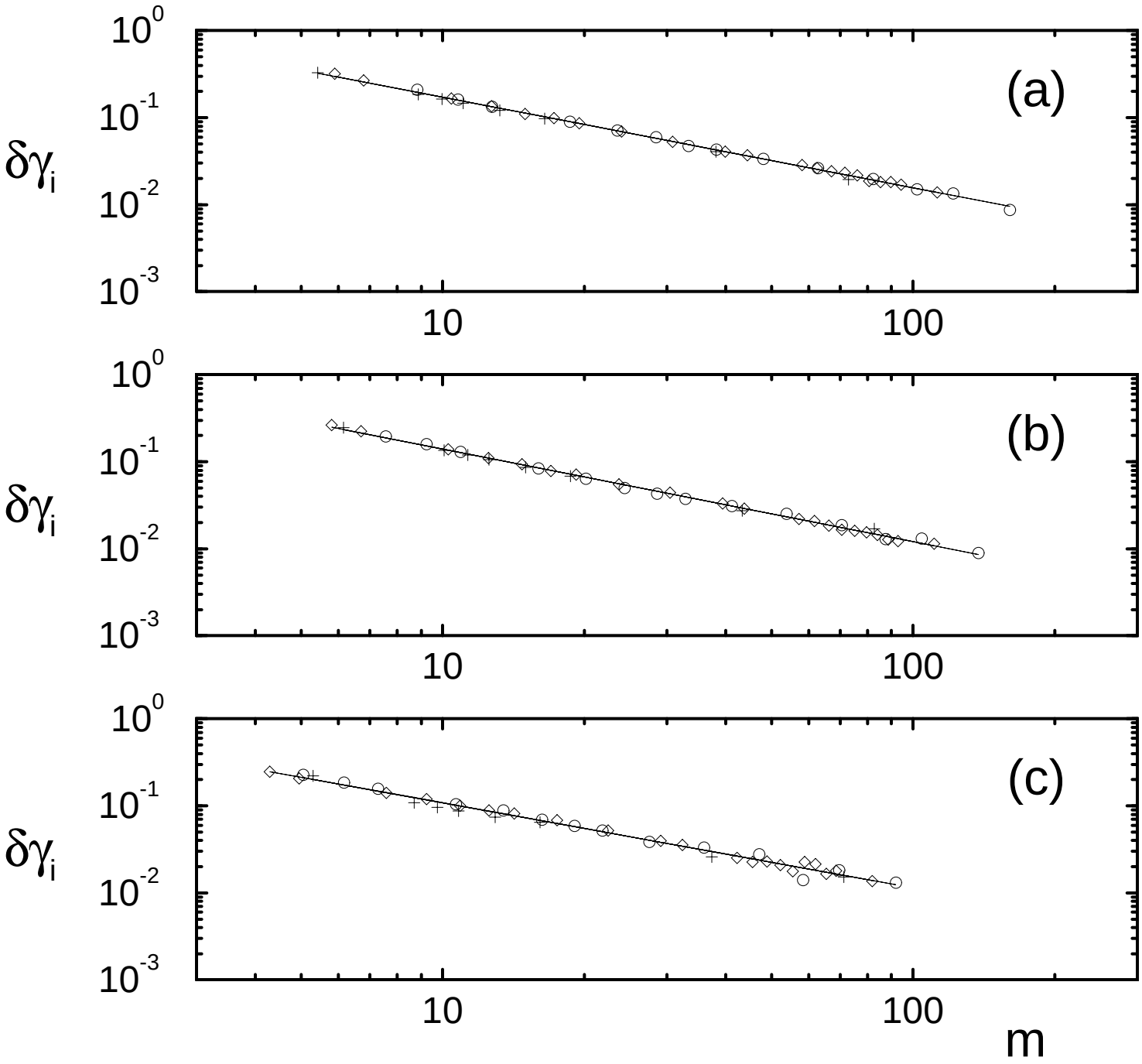


Fig.2 Kottos et al.

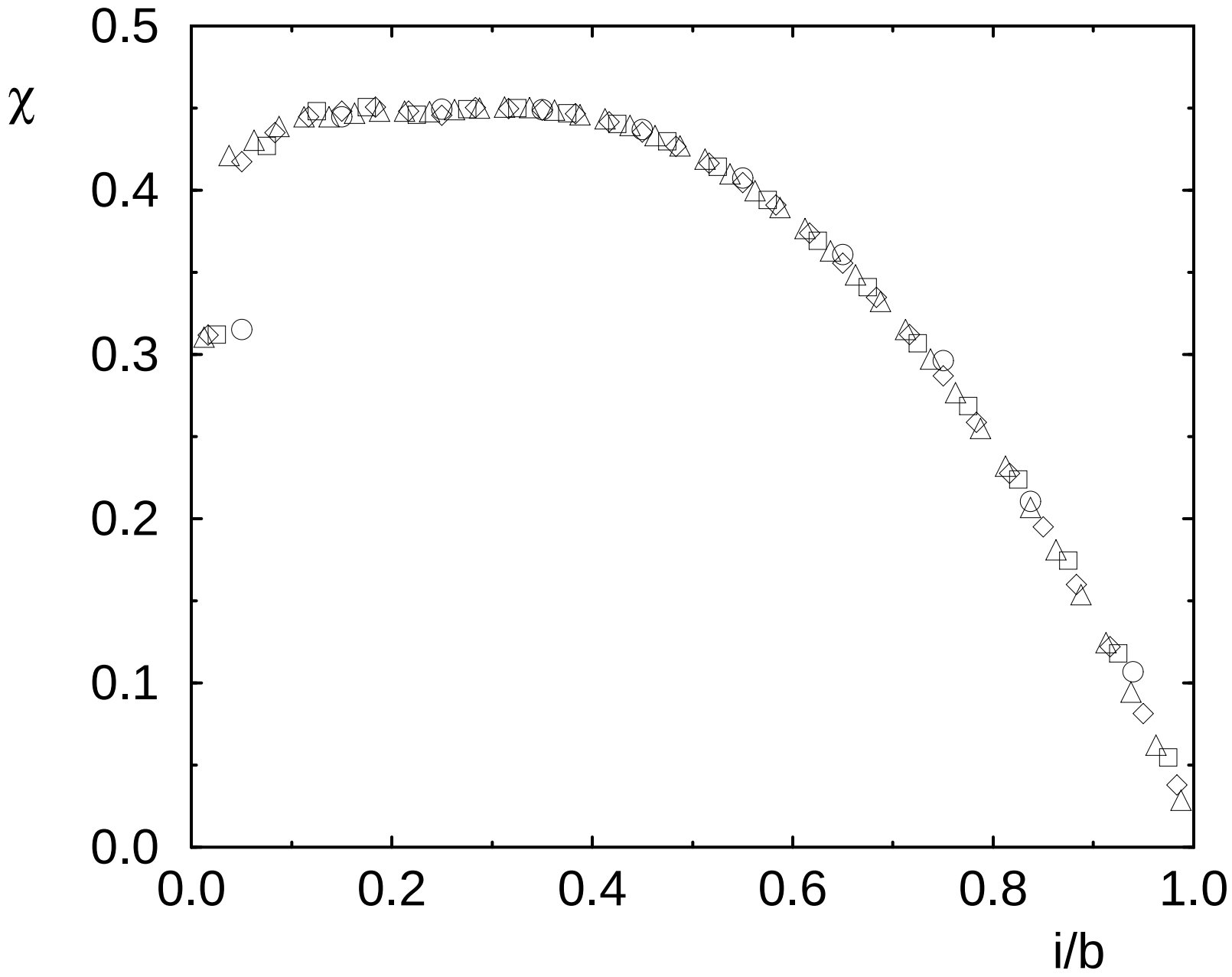


Fig.3 Kottos et al.

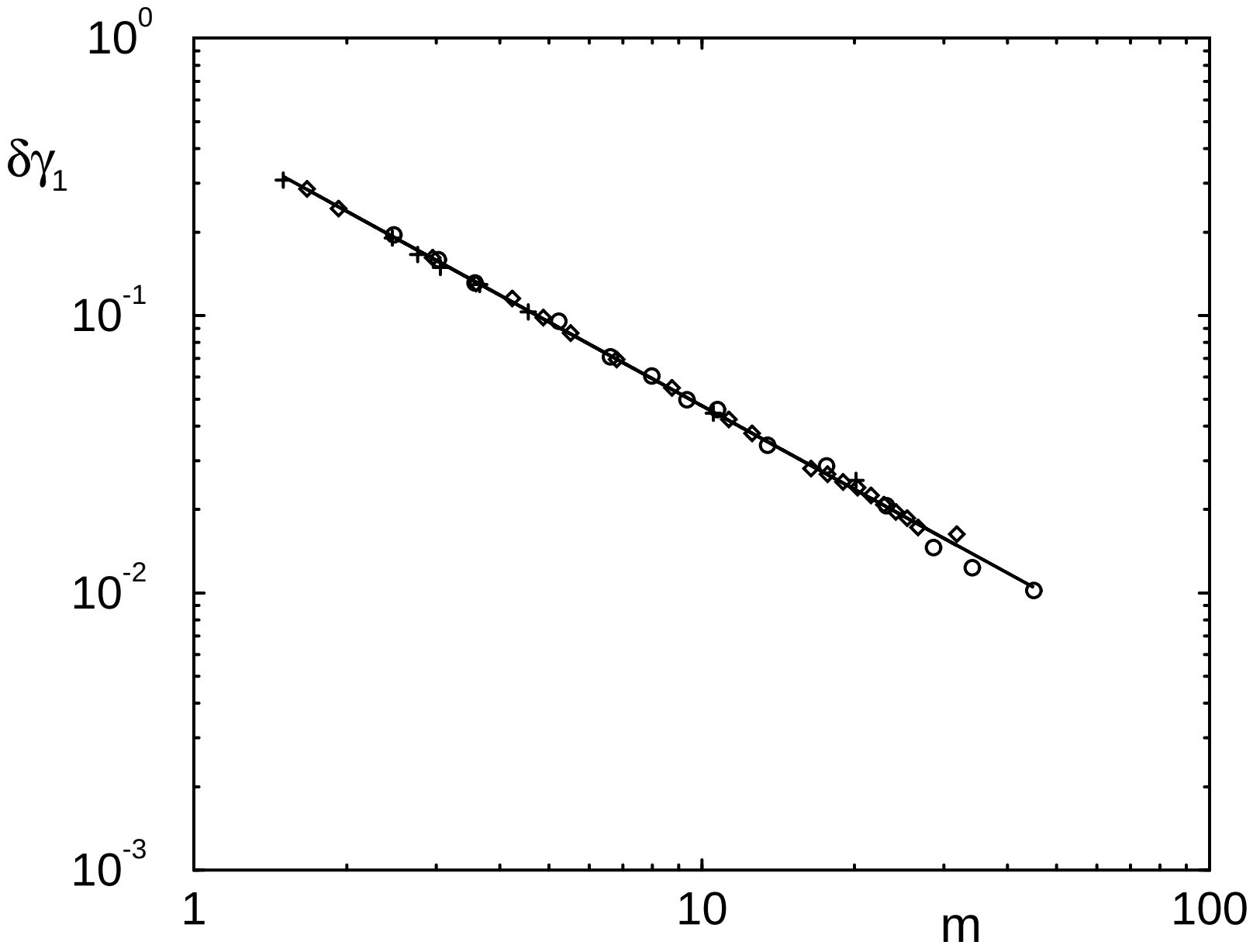


Fig.4 Kottos et al.

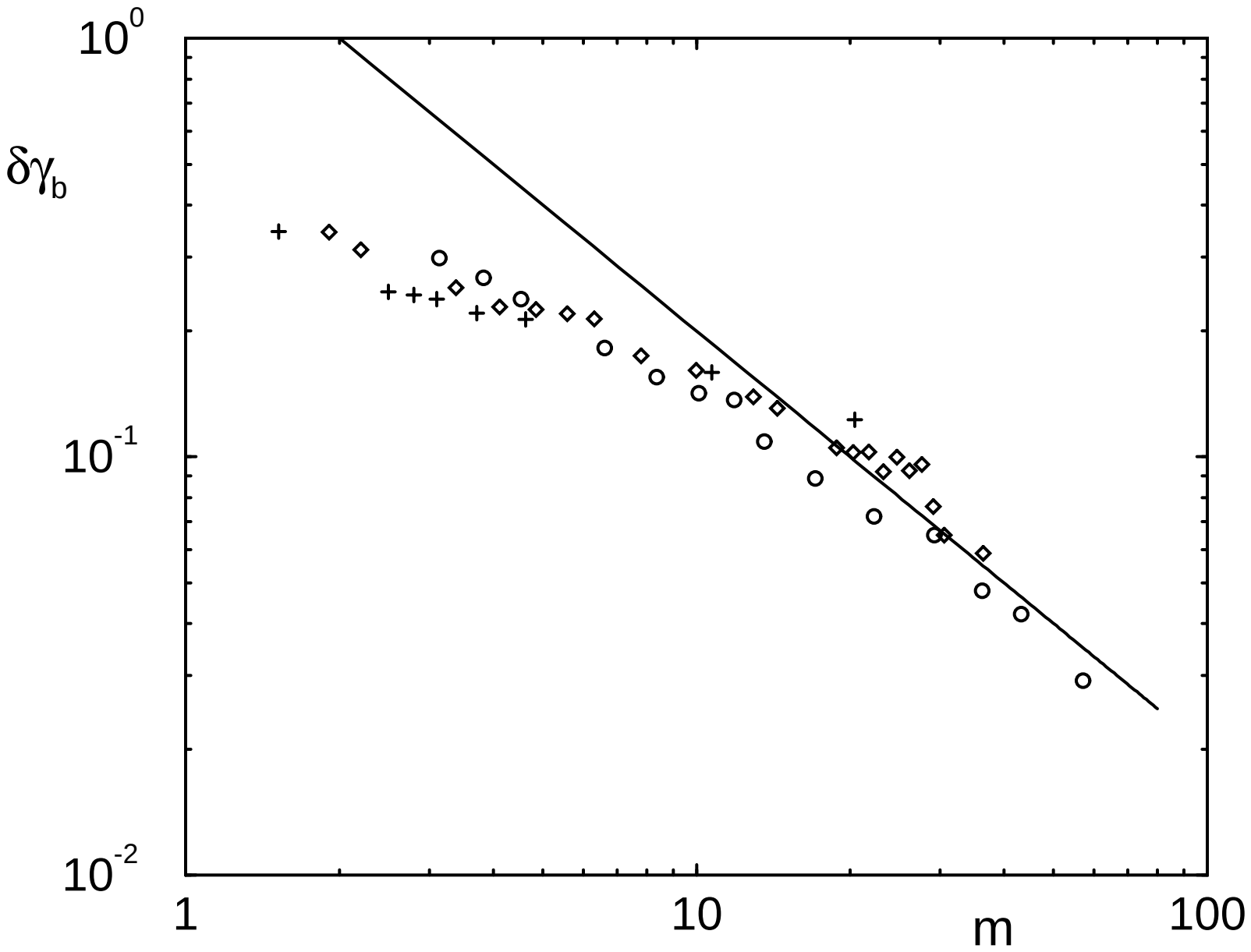


Fig.5 Kottos et al.